S-matrices*

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ABSTRACT

A new class of so called S-matrices is introduced which allows investigating links between various known classes of matrices such as Vandermonde matrices, Hankel matrices, companion matrices, etc. For complex S-matrices, the problem of decomposition into a quasidirect sum (a sum for which the sum of the ranks of the summands equals the rank of the given matrix) of indecomposable complex S-matrices is completely solved, and the uniqueness of such a decomposition is proved.

1. INTRODUCTION

We intend to investigate the class of so-called S-matrices, which comprises important classes of matrices such as Vandermonde matrices, extension of companion matrices, proper Hankel matrices, etc.

A complex matrix (in general over a field)

$$A=\begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0k} \\ a_{10} & a_{11} & \cdots & a_{1k} \\ \multicolumn{4}{c}{\dotfill} \end{pmatrix}$$

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with a finite number of columns is called an S-matrix if, for an indeterminate x, the extended matrix

$$A_x = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0k} & 1 \\ a_{10} & a_{11} & \cdots & a_{1k} & x \\ a_{20} & a_{21} & \cdots & a_{2k} & x^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

has the property that the greatest common divisor of all its subdeterminants of order $r(A)+1$, where $r(A)$ is the rank of A, is a nonzero polynomial of degree $r(A)$. This (monic) polynomial will then be called S-polynomial of A.

We shall call an S-matrix pure if its columns are linearly independent.

REMARK 1.1. In order to form subdeterminants of order $r(A)+1$, there must be at least $r(A)+1$ rows in A. Hence, any S-matrix has linearly dependent rows.

LEMMA 1.2. *An S-matrix has S-polynomial* 1 *iff it is a zero matrix.*

Proof. This follows immediately from the definition. ■

EXAMPLE 1.3. Any $m \times n$ Vandermonde matrix, $m > n$—i.e., any matrix of the form

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ t_1^2 & t_2^2 & \cdots & t_n^2 \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

with mutually distinct numbers $t_1, \dots, t_n$ and m rows—is a pure S-matrix. Its S-polynomial is $\prod_{i=1}^{n}(x - t_i)$. The same is true if the number of rows is infinite.

This follows from Theorem 2.7 as a special case.

EXAMPLE 1.4. The well-known companion matrix of a polynomial $f(x) = x^n - a_1 x^{n-1} - \cdots - a_n$ is the matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix}.$$

Its corresponding generalized companion matrix [2] is the matrix

$$C^{\infty}=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ a_n & a_{n-1} & \cdots & a_1 \\ a_n a_1 & a_n + a_1 a_{n-1} & \cdots & a_2 + a_1^2 \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

obtained in the following way: If the kth row of C^{∞} is denoted as C_k, then the first n rows $C_1,\ldots,C_n$ are identical with the first n rows of the $n\times n$ identity matrix, while for $k\geqslant n+1$,

$$C_k = a_1 C_{k-1} + a_2 C_{k-2} + \cdots + a_n C_{k-n}. \tag{1}$$

We shall show in Theorem 2.5 that C^{∞} is a pure S-matrix with S-polynomial $f(x)$.

EXAMPLE 1.5. Let t be a number. The $m\times n$ $(m>n)$ matrix

$$P_{mn}(t)=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ t & 1 & 0 & \cdots & 0 \\ t^2 & 2t & 1 & \cdots & 0 \\ t^3 & 3t^2 & 3t & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} = (p_{ij}), \tag{2}$$

$$p_{ij}=\binom{i}{j}t^{i-j}, \qquad i=0,1,\ldots,m-1, \quad j=0,1,\ldots,n-1,$$

is also a pure S-matrix. Its S-polynomial is $(x-t)^n$.

This again follows from Theorem 2.7 as a special case.

EXAMPLE 1.6. Let $A=(\alpha_{i+k})$, $i,k=0,1,\ldots,$ be an $m\times n$ proper Hankel matrix with linearly dependent rows. *Proper* means that the upper left corner submatrix of A of order $r(A)$, the rank of A, is nonsingular. Then A is an S-matrix and its S-polynomial is identical with the H-polynomial in [1].

We shall be interested in the properties of S-matrices, in particular in the so-called quasidirect decompositions of S-matrices into sums of matrices which are again S-matrices. Here, a sum $B + C$ of matrices of the same size is called quasidirect if for the ranks, $r(B + C) = r(B) + r(C)$. This definition clearly extends to quasidirect sums of more than two matrices as well as to matrices which have, as in our case, a finite number of columns but maybe an infinite number of rows.

In Example 1.3, it is clear that V is a quasidirect sum of the matrices $V_1, \dots, V_n$ of the same size as V, where V_i consists of the ith column of V, all the remaining entries being zero. The S-polynomial of V_i is then $x - t_i$.

2. RESULTS

LEMMA 2.1. *Let A, B be matrices with the same column space. Then A is an S-matrix iff B is. In such case, both matrices have the same S-polynomial.*

Proof. Follows from the fact that the definition of the S-matrix and of the S-polynomial depend on the column-space only. ■

COROLLARY 2.2. *If A is an S-matrix with m columns and G is any $m \times n$ matrix with linearly independent rows, then $B = AG$ is again an S-matrix which has, in addition, the same S-polynomial as A.*

We shall prove now:

LEMMA 2.3. *Let A be an S-matrix with n columns and rank r. Then the first r rows of A are linearly independent.*

Proof. Suppose that the rank of the submatrix of A consisting of the first r rows of A is $s < r$.

There exists a nonsingular matrix G_1 of order n such that $\hat{A} = AG_1$ has zeros in the first $n - r$ columns. By Corollary 2.2, $\hat{A}$ is again an S-matrix. Since the submatrix $\hat{A}_0$ of $\hat{A}$ consisting of the first r rows of $\hat{A}$ has also rank s, there exists a nonsingular matrix G_2 of order n such that $\hat{A}G_2$ has in the first $n - r$ columns also zeros and $\hat{A}_0 G_2$ has zeros even in the first $n - s$ columns. Thus,

$$AG_1G_2 = \begin{pmatrix} 0 & 0 & Q \\ 0 & S_1 & S_2 \end{pmatrix},$$

where Q is an $r \times s$ matrix with rank s and S_1 has $r - s$ linearly independent columns. Thus there exists a subset J_1 of $s + 1$ row indices such that the corresponding submatrix of Q has also rank s. Since S_1 has linearly independent columns, there exists a set J_2 of $r-s$ row indices such that the corresponding submatrix of S_1 is nonsingular. It follows that the submatrix of A with the last $r+1$ columns and the rows with the set of indices $J_1 \cup J_2$ has determinant which is a nonzero polynomial of degree less than r, a contradiction. ∎

COROLLARY 2.4. *The rank of an S-matrix is equal to the number r for which the first r rows of A are linearly independent whereas the first $r+1$ rows are not.*

THEOREM 2.5. *Let A be an S-matrix with rank r, and A_0 its submatrix of the first r rows. Then there exists a unique matrix $\hat{C}$ such that*

$$A = \hat{C}A_0.$$

The matrix $\hat{C}$ is a pure S-matrix, equal to the leading submatrix, of the appropriate size, of the generalized companion matrix of the S-polynomial of A.

Proof. By Lemma 2.1 and Corollary 2.2, we can assume that A with n columns is pure, $r = n$, so that A_0 is a square nonsingular matrix. Thus the matrix $M = AA_0^{-1} = (m_{ik})$, $i = 0,1,\ldots,$ $k = 1,\ldots,n$, is a pure S-matrix whose first n rows form the $n \times n$ identity matrix. Let

$$\varphi = x^n - \sum_{i=1}^{n} a_i x^{n-i}$$

be its S-polynomial, which means that the $(n+1)$st row is a linear combination of the first n rows with the coefficients $a_n,\ldots,a_1$. Denote by $\hat{C}$ the leading submatrix of the generalized companion matrix of φ of the same size as M. We shall show that $M = \hat{C}$. Let us distinguish three cases:

Case 1: $a_1 = a_2 = \cdots = a_n = 0$ so that $\varphi = x^n$. Assume there is an entry $m_{ik} \neq 0$ for $i \geqslant n$ and $1 \leqslant k \leqslant n$. Then the determinant of A_x in the first n and the $(i+1)$st row has the form $x^{i+1} - m_{ik}x^{k-1} + \cdots$ and is not divisible by x^n, a contradiction. Thus $M = \hat{C}$ in this case.

Case 2: $a_n \neq 0$. Denote, for a moment, by M_t, $t = 0,1,\ldots,$ that square submatrix of M with n rows which corresponds to row indices $t, t+1,\ldots,n+t-1$; and by m_s the row vector of M with the index s. Clearly $M_0 = I$ and M_1

is the usual companion matrix of the polynomial φ. We shall show by induction with respect to t that M_t is nonsingular for all possible values of $t = 0, 1, \dots$ and that

$$m_{n+t} = aM_t,$$

where $a = (a_n, a_{n-1}, \dots, a_1)$. Clearly, these properties characterize, together with $M_0 = I$, the matrix $\hat{C}$.

For $t = 0$, the assertion is true. Suppose thus that $t > 0$ and that the assertion is true for $t-1$. Consider the determinant

$$\det\begin{pmatrix} m_{t-1} & x^{t-1} \\ m_t & x^t \\ \cdots & \cdots \\ m_{n+t-1} & x^{n+t-1} \end{pmatrix} = x^{n+t-1}\det M_{t-1} + \cdots + (-1)^n x^{t-1}\det M_t.$$

Since it is divisible by φ and $a_n \neq 0$, $\det M_{t-1} \neq 0$ implies $\det M_t \neq 0$. Now,

$$\det\begin{pmatrix} m_t & x^t \\ \cdots & \cdots \\ m_{n+t} & x^{n+t} \end{pmatrix} = x^t \det\begin{pmatrix} M_t & X \\ m_{n+t} & x^n \end{pmatrix}$$

$$= x^t \det M_t \det\begin{pmatrix} I & X \\ m_{n+t}M_t^{-1} & x^n \end{pmatrix},$$

where $X = (1, \dots, x^{n-1})^T$. Since φ and x^t are relatively prime, we have

$$\varphi = \det\begin{pmatrix} I & X \\ m_{n+t}M_t^{-1} & x^n \end{pmatrix} = x^n - m_{n+t}M_t^{-1}X.$$

Therefore

$$m_{n+t}M_t^{-1} = a,$$

$$aM_t = m_{n+t},$$

and this together with (1) implies $M = \hat{C}$.

Case 3: $a_n = a_{n-1} = \cdots = a_{s+1} = 0$, $a_s \neq 0$, where $1 \leqslant s \leqslant n-1$. Then φ is divisible by x^{n-s} but not by x^{n-s+1}. We shall show first that M has the form

$$M = \begin{pmatrix} I_{n-s} & 0 \\ 0 & \tilde{M} \end{pmatrix} \tag{3}$$

where $\tilde{M}$ is an S-matrix with the S-polynomial $\tilde{\varphi} = x^s - a_1 x^{s-1} - \cdots - a_s$ and such that its leading submatrix with s rows is I_s. Suppose that $m_{ik} \neq 0$ for some $i \geqslant n-s$ and k satisfying $1 \leqslant k \leqslant n-s$. Then $i = n$ and, choosing the submatrix M_1 in M_x with rows $0, \ldots, n-1, i$, we obtain

$$\det M_1 = \det\begin{pmatrix} I & X \\ m_i & x^i \end{pmatrix} = x^i - m_{ik}x^{k-1} - \cdots .$$

However, this polynomial should be divisible by φ, and thus by x^{n-s}. Consequently, $k-1 \geqslant n-s$, a contradiction. Therefore, M has the form (3). The greatest common divisor of all determinants of order $n+1$ in M_x being φ, it follows easily that the greatest common divisor of all determinants of order $s+1$ in the analogous matrix $\tilde{M}_x$ is $\tilde{\varphi}$. Thus $\tilde{M}$ is an S-matrix, and by case 2, $\tilde{M}$ is a section of the generalized companion matrix of $\tilde{\varphi}$. It follows that then again $M = \hat{C}$. ∎

Corollary 2.6. *If an S-matrix A and a pure S-matrix B have the same number of rows (finite or infinite) and the same S-polynomial, then there exists a matrix M with linearly independent rows such that*

$$A = BM. \tag{4}$$

Proof. By Theorem 2.5, $A = CA_0$, $B = CB_0$ with the same matrix C, a nonsingular matrix B_0 and A_0 with linearly independent rows. Therefore, $M = B_0^{-1}A_0$ satisfies the condition (4). ∎

Definition. A *generalized Vandermonde matrix* is any matrix of the form

$$V = \left(P_{mn_1}(t_1), P_{mn_2}(t_2), \ldots, P_{mn_s}(t_s)\right) \tag{5}$$

where $t_1, \ldots, t_s$ are mutually distinct complex numbers, $n_1, n_2, \ldots, n_s$ positive integers, and $P_{mn_k}(t_k)$ matrices of the form (2),

$$m > n = \sum_{j=1}^{s} n_j.$$

THEOREM 2.7. *The matrix V in* (5) *is a pure S-matrix. Its S-polynomial is*

$$\varphi = \prod_{j=1}^{s} (x - t_j)^{n_j}. \tag{6}$$

Proof. Follows from the fact that each determinant of V_x of order $n+1$ is divisible by $(x - t_j)^{n_j}$ for each j and hence by φ, whereas such a determinant of the matrix formed by the first $n+1$ rows is equal to φ. ■

THEOREM 2.8. *Let A be an $m \times n$ S-matrix. Then the S-polynomial of A is* (6) *iff there exists a matrix M with n linearly independent rows such that*

$$A = VM \tag{7}$$

where V has the form (5).

Proof. The fact that A in (7) has S-polynomial φ follows immediately from Corollary 2.2 and Theorem 2.7. The converse is also true by Corollary 2.6. ■

Let us turn now to quasidirect decompositions of S-matrices.

THEOREM 2.9. *Let B, C be S-matrices of the same dimensions such that the sum $B + C$ is quasidirect. Then $B + C$ is again an S-matrix, and its S-polynomial is the product of the S-polynomials of B and C. In addition, these last two polynomials are relatively prime.*

Proof. Let B, C be $p \times q$. By Theorem 2.8,

$$B = V_1 M_1,$$

where M_1 is $r(B) \times q$ of rank $r(B)$, and

$$C = V_2 M_2$$

where M_2 is $r(C) \times q$ of rank $r(C)$, and V_1, V_2 are matrices of the form (5).

Consequently, we can write

$$B + C = (V_1, V_2)\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}. \tag{8}$$

Since $B + C$ is quasidirect, both factors on the right hand side have rank

$r(B)+r(C)$. This means, however, that $B+C$ has the form (7) and (V_1, V_2) is again a generalized Vandermonde matrix with distinct numbers in V_1 and V_2. By Theorem 2.8, $B+C$ is an S-matrix whose S-polynomial is the product of the S-polynomials of B and C, and these polynomials are relatively prime. ■

We shall be able to solve completely the problem of decomposition of S-matrices into quasidirect sum of S-matrices. We say that an S-matrix A is S-indecomposable if no quasidirect decomposition $A = B + C$ exists where B and C are nonzero S-matrices. This notion depends, however, on the field, as the example of the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with S-polynomial x^2+1 shows. For simplicity, we shall investigate the case of the complex field only.

THEOREM 2.10. *A complex S-matrix is S-indecomposable over the field of complex numbers iff its S-polynomial is a nonnegative power of a linear polynomial.*

Every complex S-matrix A is a quasidirect sum of S-indecomposable S-matrices, and this sum is unique up to the ordering of the summands. Each summand corresponds to one root of the S-polynomial of A.

Proof. The "if" part of the first assertion follows from Theorem 2.9 and Lemma 1.2. Let now A be an $m \times n$ S-matrix whose S-polynomial φ is not such a power. Then

$$\varphi = \varphi_1\varphi_2 \cdots \varphi_s, \qquad \varphi_i = (x-t_i)^{n_i}, \quad n_i = 1, \; i = 1, \ldots, s,$$

t_i mutually distinct, $s > 1$, $\sum_i n_i = n$. Let V be the generalized Vandermonde matrix formed as in (5), let $V_i = P_{mn_i}(t_i)$. By Theorem 2.7, V is a pure S-matrix with the S-polynomial φ. By Corollary 2.6,

$$A = VM$$

for some matrix M with linearly independent rows. Let

$$M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_s \end{pmatrix}$$

be the partitioning of M corresponding to the partitioning

$$V = (V_1, V_2, \ldots, V_s).$$

Then

$$A = \sum_{i=1}^{s} A_i, \tag{9}$$

where

$$A_i = V_i M_i$$

are, by Theorem 2.8, S-matrices with S-polynomials φ_i, $i = 1, \ldots, s$. These matrices are S-indecomposable. The ranks of A_i being n_i and the rank of A being $n = \sum_{i=1}^{s} n_i$, (9) is quasidirect. This completes the proof of the first part and of the existence of a decomposition. To prove the uniqueness, assume that

$$A = \sum_{j=1}^{v} B_j$$

is also a quasidirect decomposition of A into S-indecomposable S-matrices. By the previous part, each B_j corresponds to one root of the S-polynomial of A. Therefore, $v = s$ and we can assume that B_j corresponds to t_j. It follows that

$$B_j = V_j N_j, \qquad j = 1, \ldots, s,$$

where $V_j = P_{mn_j}(t_j)$ as above. Thus,

$$A = VN, \qquad N = \begin{pmatrix} N_1 \\ \vdots \\ N_s \end{pmatrix}.$$

Let $\hat{A}, \hat{V}$ be the submatrices of A, V consisting of the first n rows. Then $\hat{A} = \hat{V}M$ as well as $\hat{A} = \hat{V}N$. Since $\hat{V}$ is nonsingular, $M = N$. ∎

REMARK 2.11. A nonsingular matrix can also be written as a sum of S-matrices. It suffices to complete such a matrix by another row. The resulting matrix is an S-matrix and by Theorem 2.10 can be decomposed into a sum of S-indecomposable S-matrices. If there are at least two summands, we obtain, by leaving out the last rows in each summand, a quasidirect decomposition of the original matrix into a sum of S-matrices. This decomposition is, of course, not unique if the order of the given matrix is greater than one.

We shall conclude with two corollaries of Theorem 2.10.

COROLLARY 2.12. *Let A be a proper Hankel matrix with linearly dependent rows. If $A=\sum_i A_i$ is a quasidirect decomposition of A into a sum of S-matrices, then all matrices A_i are again proper Hankel matrices.*

Proof. We shall need a result of [1] which states that a proper complex Hankel matrix A can be decomposed into a quasidirect sum of H-indecomposable proper Hankel matrices, each of the summands corresponding to one root of the H-polynomial of A. Since the H-polynomial of A is identical with the S-polynomial of A, this decomposition is also the unique decomposition of A into a sum of proper Hankel matrices with relatively prime H-polynomials, again a proper Hankel matrix. ■

COROLLARY 2.13. *Let $f(x)$, $g(x)$ be polynomials of degrees m,n respectively. Let $\hat{C}_f,\hat{C}_g$ be the leading sections of the generalized companion matrices $C_f^{\infty},C_g^{\infty}$ of the polynomials f,g, each having $m+n$ rows. Then f and g are relatively prime iff*

$$\det\left(\hat{C}_f,\hat{C}_g\right)\neq 0.$$

Proof. By Theorem 2.8, $\hat{C}_f=V_fM_f$, $\hat{C}_g=V_gM_g$, where V_f,V_g are the generalized Vandermonde matrices with $m+n$ rows corresponding to f,g respectively and M_f, M_g are nonsingular matrices. Since

$$\left(\hat{C}_f,\hat{C}_g\right)=\left(V_f,V_g\right)\begin{pmatrix} M_f & 0\\ 0 & M_g\end{pmatrix}$$

and (V_f,V_g) is nonsingular iff f and g are relatively prime, the result follows. ■

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